ALGEBRAIC EIGEN VALUE PROBLEMS

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$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

square matrix

unknown vector

unknown scalar

$\underline{x} = \underline{0}$: (no practical interest)

$\underline{x} \neq \underline{0}$: eigenvectors of **A**; exist only for certain values of λ (eigenvalues or characteristic roots)

→ Multiplication of **A** = same effect as the multiplication of **x** by a scalar λ

- Eigenvalue: special set of scalars associated with a linear systems of equations. Each eigenvalue is paired with a corresponding eigenvectors.

Eigenvalues, Eigenvectors

- Eigenvalue problems: $\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$ or $\left(\underline{\underline{A}} - \lambda\underline{\underline{I}}\right)\underline{x} = \underline{0}$

eigenvectors

eigenvectors

Set of eigenvalues: spectrum of A

Largest of $|\lambda_i|$: spectral radius of A

How to Find Eigenvalues and Eigenvectors

Ex. 1.)

$$\underline{\underline{A}} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 \end{aligned} \qquad \left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{x} = \underline{0}$$

In homogeneous linear system, nontrivial solutions exist when det (**A**-λ**I**)=0.

Characteristic equation of **A**:

$$D(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = 0$$

Characteristic polynomial

Characteristic determinant

Eigenvalues: λ_1=-1 and λ_2= -6

Eigenvectors: for λ_1=-1, $\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for λ_2=-6, $\underline{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

obtained from Gauss elimination

General Case

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= \lambda x_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

$$\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{x} = \underline{0}, \quad D(\lambda) = \det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = 0$$

Theorem 1:

Eigenvalues of a square matrix **A** → roots of the characteristic equation of **A**.
nxn matrix has at least one eigenvalue, and at most n numerically different eigenvalues.

Theorem 2:

If $\underline{x}$ is an eigenvector of a matrix **A**, corresponding to an eigenvalue λ,
so is $k\underline{x}$ with any $k \neq 0$.

Ex. 2) multiple eigenvalue

- Algebraic multiplicity of λ: order M_λ of an eigenvalue λ
  Geometric multiplicity of λ: number of m_λ of linear independent eigenvectors
  corresponding to λ. (=dimension of eigenspace of λ)

  In general, $m_\lambda \leq M_\lambda$

  Defect of λ: $\Delta_\lambda = M_\lambda - m_\lambda$

Ex 3) algebraic & geometric multiplicity, positive defect

Ex. 4) complex eigenvalues and eigenvectors

Some Applications of Eigenvalue Problems

Ex. 1) Stretching of an elastic membrane.

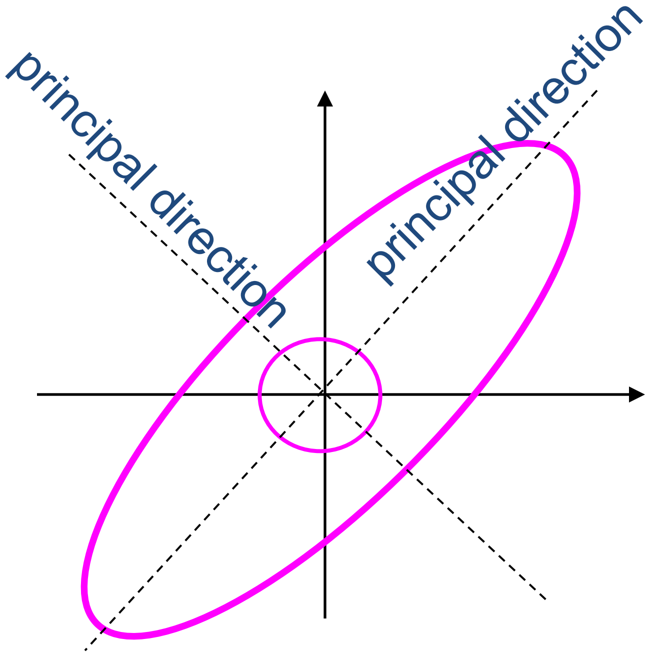
Find the principal directions: direction of position vector $\underline{x}$ of P

= (same or opposite) direction of the position vector $\underline{y}$ of Q

$$x_1^2 + x_2^2 = 1,\ \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{y} = \underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad \Rightarrow \lambda_1 = 8,\ \underline{x}_1 \text{ for } \lambda_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2,\ \underline{x}_2 \text{ for } \lambda_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



Eigenvalue represents speed of response
Eigenvector ~ direction

Ex. 4) Vibrating system of two masses on two springs

$$y_1'' = -5y_1 + 2y_2$$

$$y_2'' = 2y_1 - 2y_2$$

Solution vector: $\underline{y} = \underline{x}e^{wt}$

$\Rightarrow \underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad (\lambda = w^2)$ solve eigenvalues and eigenvectors

$$\Rightarrow \underline{y} = \underline{x}_1(a_1 \cos t + b_1 \sin t) + \underline{x}_2(a_2 \cos\sqrt{6}t + b_2 \sin\sqrt{6}t)$$

Examples for stability analysis of linear ODE systems using eigenmodes

Stability criterion: signs of real part of eigenvalues of the matrix

$$\dot{\underline{x}} = \frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

A determine the stability of the linear system.

Re(λ) < 0: stable

Re(λ) > 0: unstable

Ex. 1) Node-sink

$$\dot{x}_1 = -0.5x_1 + x_2 \quad \Rightarrow \lambda_1 = -0.5$$

$$\dot{x}_2 = -2x_2 \qquad\qquad \lambda_2 = -2$$

stable

Ex. 2) Saddle

$$\dot{x}_1 = 2x_1 + x_2 \qquad \Rightarrow \lambda_1 = -1.5616, \underline{x}_1 \text{ for } \lambda_1 = \begin{pmatrix} 0.2703 \\ -0.9628 \end{pmatrix}$$

$$\dot{x}_2 = 2x_1 - x_2$$

unstable

$$\lambda_2 = 2.5616, \underline{x}_2 \text{ for } \lambda_2 = \begin{pmatrix} 0.8719 \\ 0.4896 \end{pmatrix}$$

Phase plane ?

Ex. 3) Unstable focus

$$\dot{x}_1 = x_1 + 2x_2 \qquad \Rightarrow \lambda_1 = 1 + 2i$$

$$\dot{x}_2 = -2x_1 + x_2 \qquad \lambda_2 = 1 - 2i$$

unstable

Phase plane ?

Ex. 4) Center

$$\dot{x}_1 = -x_1 - x_2 \qquad \Rightarrow \lambda_1 = 0 + 1.7321i$$

$$\dot{x}_2 = 4x_1 + x_2 \qquad \lambda_2 = 0 - 1.7321i$$

Properties of Eigenvalue

1) Trace $A = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

2) det $A = \prod_{i=1}^{n} \lambda_i$

3) If A is symmetric, then the eigenvectors are orthogonal:

$$x_i^T x_j = \begin{cases} 0, & i \neq j \\ G_{ii} & i = j \end{cases}$$

4) Let the eigenvalues of $A = \lambda_1, \lambda_2, \cdots \lambda_n$ then, the eigenvalues of (*A* - *aI*)

$$= \lambda_1 - a, \lambda_2 - a, \cdots, \lambda_n - a,$$

- Transformation Ax

  $Ax = \lambda x$: The transformation of an eigenvector is mapped onto the same line of.

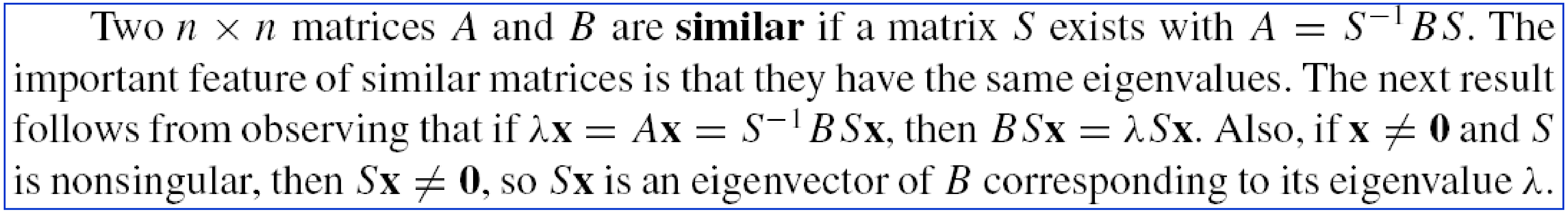
Geometrical Interpretation of Eigenvectors

- Symmetric matrix → orthogonal eigenvectors

- Relation to Singular Value

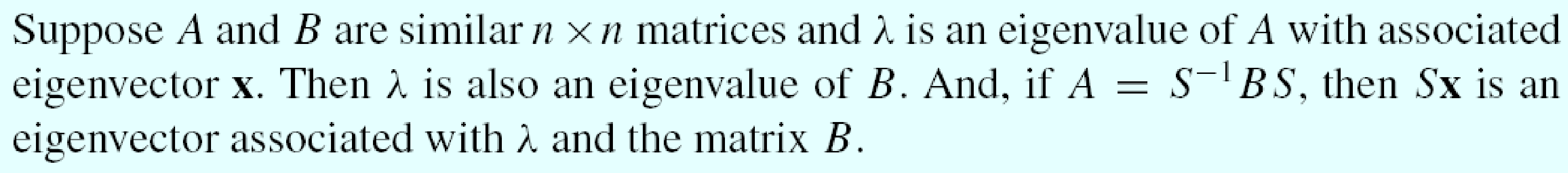
  if ***A*** is singular → 0 $\in$ {eigenvalues}

Similar Matrices



Two $n \times n$ matrices A and B are **similar** if a matrix S exists with $A = S^{-1}BS$. The important feature of similar matrices is that they have the same eigenvalues. The next result follows from observing that if $\lambda\mathbf{x} = A\mathbf{x} = S^{-1}BS\mathbf{x}$, then $BS\mathbf{x} = \lambda S\mathbf{x}$. Also, if $\mathbf{x} \neq \mathbf{0}$ and S is nonsingular, then $S\mathbf{x} \neq \mathbf{0}$, so $S\mathbf{x}$ is an eigenvector of B corresponding to its eigenvalue λ.

- Eigenvalues and eigenvectors of similar matrices



Suppose A and B are similar $n \times n$ matrices and λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$. Then λ is also an eigenvalue of B. And, if $A = S^{-1}BS$, then $S\mathbf{x}$ is an eigenvector associated with λ and the matrix B.

Eg. Rotation matrix

contents

Jacobi’s Method

- Requires a symmetric matrix
- May take numerous iterations to converge
- Also requires repeated evaluation of the arctan function

- Isn’t there a better way?
- Yes, but we need to build some tools.

Jacobi method idea

Find large off-diagonal element $a_{p,q}$.

Set it to zero using $R_{p,q}(\theta)$.

Since $\tilde{a}_{p,q}$=($a_{p,q}$ - $a_{p,p}$)sinθcosθ+$a_{p,q}$($\cos^2$ ø – $\sin^2$ ø),

Set

$$\tan 2\theta = \frac{2a_{p,q}}{a_{p,p-}a_{q,q}}$$

The Jacobi Algorithm

- One of the oldest numerical methods, but still of interest
- Start with initial guess at V (e.g. V=I), set $A=V^TAV$
- For k=1, 2, …
  - If A is close enough to diagonal (Frobenius norm of off-diagonal tiny relative to A) stop
  - Find a Givens rotation Q that solves a 2x2 subproblem
    - Either zeroing max off-diagonal entry, or sweeping in order through matrix.
  - $V=VQ$, $A=Q^TAQ$
- Typically O(log n) sweeps, $O(n^2)$ rotations each
- Quadratic convergence in the limit

- Example

$$A = \begin{pmatrix} 1 & 1 & 8 \\ 1 & 2 & 2 \\ 8 & 2 & 1 \end{pmatrix}$$

- pick another big off-diagonal element: repeat.
- PROBLEM: zero elements are made nonzero
- again.

Givens method idea

- Take off-diagonal element $a_{p,q,}\ |p-q| \geq 2$
- Set it to zero using

since $R_{p+1,q}(\theta)$

set $$\overset{\square}{a}_{p,q} = -a_{p,p+1}\sin\theta + a_{p,q}\cos\theta,$$

$$\tan\theta = \frac{a_{p,q}}{a_{p,p+1}}$$

- Example

$$A = \begin{pmatrix} 1 & 8 & 8 & 3 \\ 8 & 2 & 5 & 4 \\ 8 & 5 & 3 & 1 \\ 3 & 4 & 1 & 1 \end{pmatrix}$$

- Work through elements in turn
- (p, q) = (1, 3), (1, 4), . . . , (1, n), (2, 4), (2, 5),. . .(2, n), (3, 1),.. (n − 3, n − 1), (n − 3, n),….(n − 2, n).
- PROBLEM: This cannot reduce beyond tridiagonal form.

Givens Method

- – based on plane rotations
- – reduces to tridiagonal form
- – finite (direct) algorithm

What Householder’s Method Does

- Preprocesses a matrix **A** to produce an upper-Hessenberg form **B**
- The eigenvalues of **B** are related to the eigenvalues of **A** by a linear transformation
- Typically, the eigenvalues of **B** are easier to obtain because the transformation simplifies computation

House holder method

- – based on an orthogonal transformation
- – reduces to tridiagonal form
- – finite (direct) algorithm

Definition: Upper-Hessenberg Form

- A matrix B is said to be in upper-Hessenberg form if it has the following structure:

$$\mathbf{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,n-1} & b_{2,n} \\ 0 & b_{3,2} & b_{3,3} & \cdots & b_{3,n-1} & b_{3,n} \\ 0 & 0 & b_{4,3} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & \cdots & 0 & b_{n,n-1} & b_{n,n} \end{bmatrix}$$

A Useful Matrix Construction

- Assume an n x 1 vector $\mathbf{u} \neq \mathbf{0}$
- Consider the matrix $\mathbf{P}(\mathbf{u})$ defined by
  $\mathbf{P}(\mathbf{u}) = \mathbf{I} - 2(\mathbf{u}\mathbf{u}^T)/(\mathbf{u}^T\mathbf{u})$
- Where
  - $\mathbf{I}$ is the n x n identity matrix
  - $(\mathbf{u}\mathbf{u}^T)$ is an n x n matrix, the *outer product* of **u** with its transpose
  - $(\mathbf{u}^T\mathbf{u})$ here denotes the trace of a 1 x 1 matrix and is the *inner or dot product*

Properties of P(u)

- $\mathbf{P}^2(\mathbf{u}) = \mathbf{I}$
  - The notation here $\mathbf{P}^2(\mathbf{u}) = \mathbf{P}(\mathbf{u}) * \mathbf{P}(\mathbf{u})$
  - Can you show that $\mathbf{P}^2(\mathbf{u}) = \mathbf{I}$?
- $\mathbf{P}^{-1}(\mathbf{u}) = \mathbf{P}(\mathbf{u})$
  - **P**(**u**) is its own inverse
- $\mathbf{P}^T(\mathbf{u}) = \mathbf{P}(\mathbf{u})$
  - **P**(**u**) is its own transpose
  - Why?
- **P**(**u**) is an orthogonal matrix

Householder's Algorithm

- Set **Q** = **I**, where **I** is an n x n identity matrix
- For k = 1 to n-2
  - α = sgn($A_{k+1,k}$)sqrt(($A_{k+1,k})^2$+ ($A_{k+2,k})^2$+…+ ($A_{n,k})^2$)
  - $\mathbf{u}^T = [0, 0, \ldots, A_{k+1,k} + \alpha, A_{k+2,k}, \ldots, A_{n,k}]$
  - $\mathbf{P} = \mathbf{I} - 2(\mathbf{u}\mathbf{u}^T)/(\mathbf{u}^T\mathbf{u})$
  - **Q = QP**
  - **A = PAP**
- Set **B** = **A**

Example

$$\text{Let } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 4 & 8 & 9 \end{bmatrix}_{3x3} .$$

$\text{Clearly, n} = 3 \text{ and since n - } 2 = 1, \text{k takes only the value k} = 1.$

$$\text{Then } \alpha = \text{sgn}(\text{a}_{21})\sqrt{a_{21}^2 + a_{31}^2} = \text{sgn}(3)\sqrt{3^2 + 4^2} = 1 \cdot 5 = 5$$

Example

$$\mathbf{u}^T = [0,\ldots,0,a_{21}+\alpha,a_{31},\ldots,a_{n1}] = [0,a_{21}+\alpha,a_{31}]$$

$$= [0,3+5,4] = [0,8,4]$$

$$\mathbf{P} = \mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\frac{\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}\begin{bmatrix} 0 & 8 & 4 \end{bmatrix}}{\begin{bmatrix} 0 & 8 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}} = ?$$

Example

$$\mathbf{P} = \mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\frac{\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}\begin{bmatrix} 0 & 8 & 4 \end{bmatrix}}{\begin{bmatrix} 0 & 8 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{80}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 64 & 32 \\ 0 & 32 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix}. \text{Find } \mathbf{P}^2 = ?$$

Example

$$\text{Initially}, \mathbf{Q} = \mathbf{I}, \text{so}$$

$$\mathbf{Q} = \mathbf{QP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$$

Example

Next, $\mathbf{A} = \mathbf{PAP}$, so

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 4 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-3}{5} & \frac{-4}{5} \\ 0 & \frac{-4}{5} & \frac{3}{5} \end{bmatrix} = ?$$

Example

Hence,

$$\mathbf{A}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{-3}{5} & \frac{-4}{5}\\ 0 & \frac{-4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 3 & 5 & 6\\ 4 & 8 & 9\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{-3}{5} & \frac{-4}{5}\\ 0 & \frac{-4}{5} & \frac{3}{5}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ -5 & \frac{-47}{5} & \frac{-54}{5}\\ 0 & \frac{4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{-3}{5} & \frac{-4}{5}\\ 0 & \frac{-4}{5} & \frac{3}{5}\end{bmatrix}$$

$$=\begin{bmatrix}1 & \frac{-18}{5} & \frac{1}{5}\\ -5 & \frac{357}{25} & \frac{26}{25}\\ 0 & \frac{-24}{25} & \frac{-7}{25}\end{bmatrix}$$

Example

Finally, since the loop only executes once

$$\mathbf{B} = \mathbf{A} = \begin{bmatrix} 1 & \frac{-18}{5} & \frac{1}{5} \\ -5 & \frac{357}{25} & \frac{26}{25} \\ 0 & \frac{-24}{25} & \frac{-7}{25} \end{bmatrix}.$$

So what?

How Does It Work?

- Householder's algorithm uses a sequence of similarity transformations
  B = **P**(**u**^{k}) **A P**(**u**^{k})
  to create zeros below the first sub-diagonal
- **u**^{k}=[0, 0, …, $A_{k+1,k}+\alpha$, $A_{k+2,k}$,…, $A_{n,k}$]^T
- α = sgn($A_{k+1,k}$)sqrt(($A_{k+1,k}$)^2+ ($A_{k+2,k}$)^2+…+ ($A_{n,k}$)^2)
- By definition,
  - sgn(x) = 1, if x≥0 and
  - sgn(x) = -1, if x<0

How Does It Work? (continued)

- The matrix **Q** is orthogonal
  - the matrices **P** are orthogonal
  - **Q** is a product of the matrices **P**
  - The product of orthogonal matrices is an orthogonal matrix
- $\mathbf{B} = \mathbf{Q}^T \mathbf{A} \mathbf{Q}$ hence $\mathbf{Q} \mathbf{B} = \mathbf{Q} \mathbf{Q}^T \mathbf{A} \mathbf{Q} = \mathbf{A} \mathbf{Q}$
  - $\mathbf{Q} \mathbf{Q}^T = \mathbf{I}$ (by the orthogonality of **Q**)

How Does It Work? (continued)

- If $\mathbf{e}^k$ is an eigenvector of **B** with eigenvalue λ_k, then $\mathbf{B}\,\mathbf{e}^k = \lambda_k\,\mathbf{e}^k$
- Since **Q B = A Q,**
  $\mathbf{A}\,(\mathbf{Q}\,\mathbf{e}^k) = \mathbf{Q}\,(\mathbf{B}\,\mathbf{e}^k) = \mathbf{Q}\,(\lambda_k\,\mathbf{e}^k) = \lambda_k\,(\mathbf{Q}\,\mathbf{e}^k)$
- Note from this:
  - λ_k is an eigenvalue of **A**
  - $\mathbf{Q}\,\mathbf{e}^k$ is the corresponding eigenvector of **A**

The Power Method

- Start with some random vector v, $||v||_2=1$
- Iterate $v=(Av)/||Av||$
- What happens? How fast?

The QR Method: Start-up

- Given a matrix **A**
- Apply Householder's Algorithm to obtain a matrix **B** in upper-Hessenberg form
- Select $\varepsilon>0$ and m>0
  - ε is a acceptable proximity to zero for sub-diagonal elements
  - m is an iteration limit

The QR Method: Main Loop

```
Do{
  Set $\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$
  For k = 1 to n -1{
    $c = \frac{B_{k,k}}{\sqrt{B_{k,k}^2 + B_{k+1,k}^2}}$ ;  $s = \frac{B_{k+1,k}}{\sqrt{B_{k,k}^2 + B_{k+1,k}^2}}$;
    Set $\mathbf{P} = \mathbf{I}$; $\mathrm{P}_{\mathrm{k,k}} = \mathrm{P}_{\mathrm{k+1,k+1}} = \mathrm{c}$;  $\mathrm{P}_{\mathrm{k+1,k}} = -\mathrm{P}_{\mathrm{k,k+1}} = -\mathrm{s}$;
    $\mathbf{B} = \mathbf{PB}$;
    $\mathbf{Q}^{\mathrm{T}} = \mathbf{PQ}^{\mathrm{T}}$;
  }
  $\mathbf{B} = \mathbf{BQ}$;
  i++;
} While ($\mathbf{B}$ is not upper block triangular) and (i < m)
```

The QR Method: Finding The λ's

Since $\mathbf{B}$ is upperblock triangular, one may compute λ_{k} from the diagonal blocks of $\mathbf{B}$. Specifically, the eigvalues of $\mathbf{B}$ are the eigenvalues of its diagonal blocks $\mathbf{B}_{\mathrm{k}}$.

If a diagonal block $\mathbf{B}_k$ is 1x1, i.e., $\mathbf{B}_k = [a]$, then $\lambda_k = a$.

If a diagonal block $\mathbf{B}_k$ is 2x2, i.e., $\mathbf{B}_k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

then $\lambda_{k,k+1} = \dfrac{trace(\mathbf{B}_k) \pm \sqrt{trace^2(\mathbf{B}_k) - 4\det(\mathbf{B}_k)}}{2}$.

Details Of The Eigenvalue Formulae

$$\text{Suppose}\,\mathbf{B}_k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$\lambda\mathbf{I} - \mathbf{B}_k = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$$

$$\left|\lambda\mathbf{I} - \mathbf{B}_k\right| = ?$$

Details Of The Eigenvalue Formulae

$$\text{Given } \lambda\mathbf{I} - \mathbf{B}_k = \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}$$

$$\begin{aligned} \left|\lambda\mathbf{I} - \mathbf{B}_k\right| &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc \\ &= \lambda^2 - trace(\mathbf{B}_k)\lambda + \det(\mathbf{B}_k) \end{aligned}$$